

(CO)HOMOLOGY OF LIE ALGEBRAS VIA ALGEBRAIC MORSE THEORY 1

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ABSTRACT. The fundamental theorem of cancellation [4] and [8], which is the algebraic generalization of discrete Morse theory [2] for simplicial complexes, is discussed in the context of general chain complexes of free modules.

Using this, we compute the Chevalley (co)homology of the Lie algebra of all triangular matrices \mathfrak{sol}_n over \mathbb{Q} or \mathbb{Z}_p for large enough primes p . We determine the column and row in the table of $H_k(\mathfrak{sol}_n; \mathbb{Z})$ where the p -torsion first appears. Every \mathbb{Z}_{p^k} appears as a direct summand of some $H_k(\mathfrak{sol}_n; \mathbb{Z})$. Module $H_k(\mathfrak{sol}_n; \mathbb{Z}_p)$ is expressed by the homology of a chain subcomplex for the Lie algebra of all strictly triangular matrices \mathfrak{nil}_n , using the Künneth formula. All conclusions are accompanied by computer experiments.

Then we generalize some results to Lie algebras of (strictly) triangular matrices \mathfrak{gl}_n^{\prec} and $\mathfrak{gl}_n^{\preceq}$ with respect to any partial ordering \preceq on $[n]$. We determine the multiplicative structure of $H^*(\mathfrak{gl}_n^{\preceq})$ w.r.t. the cup product over fields of zero or sufficiently large characteristic, the result being the exterior algebra.

Matchings used here can be analogously defined for other Lie algebra families and in other (co)homology theories; we collectively call them *normalization matchings*. They are useful for theoretical as well as computational purposes.

Background. Algebraic Morse Theory, abbreviated by AMT, is a combinatorial approach to Homological Algebra, designed for dealing with (co)chain complexes. In principle, it can be applied to any chain complex of free modules, no matter its origin, though finding a meaningful Morse matching is often difficult. The theory was discovered independently in the article [8] and thesis [4], building on the ideas of Forman's [2] in a much broader scope.

There are many applications of Forman's DMT, but using AMT happened more rarely up to now. Sköldberg used AMT to compute the Chevalley homology $H_*(\mathfrak{h}_n; R)$ of the Heisenberg Lie algebra family over any ring of characteristic 2 in [9]. Jöllenbeck applied AMT to obtain the Hochschild homology $HH_*(A; R)$ of a very general family of commutative and noncommutative R -algebras in [5, 6].

The theory of AMT lies at the intersection of mathematical fields of homological algebra, algebraic topology, algebraic combinatorics, and commutative algebra.

Results. In this article, we define a class of Morse matchings 3.5 5.3, which drastically reduce the size of chain complexes without changing the boundary; they are easily implementable. By inspecting the unmatched basis elements, we prove the following assertions. Let \preceq be any partial ordering of $[n]$. Let \mathfrak{gl}_n^{\prec} and $\mathfrak{gl}_n^{\preceq}$ be Lie subalgebras of \mathfrak{gl}_n with bases $\{e_{ij}; i \preceq j\}$ and $\{e_{ij}; i \prec j\}$ respectively.

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If R is a field of characteristic 0 or p (or more generally if $\mathbb{Q} \leq R$ or $\mathbb{Z}_p \leq R$) such that $p \geq n$, then 5.10 $H^*(\mathfrak{gl}_n^{\preceq}(R)) \cong \Lambda_R[x_1, \dots, x_n]$ as graded algebras, where $\deg x_i = 1$. If $p > \frac{k+1}{2}$, 5.8 then $H_k(\mathfrak{gl}_n^{\preceq}; \mathbb{Z})$ does not contain p -torsion. If $R = \mathbb{Z}$, then 5.5 $H_1(\mathfrak{gl}_n^{\preceq}) \cong H^1(\mathfrak{gl}_n^{\preceq}) \cong R^n$, $H_2(\mathfrak{gl}_n^{\preceq}) \cong H^2(\mathfrak{gl}_n^{\preceq}) \cong R^{\binom{n}{2}}$, $H_3(\mathfrak{gl}_n^{\preceq}) \cong R^{\binom{n}{3}} \oplus (\frac{R}{2R})^{|\{a,c \in [n]; \exists b: a \prec b \prec c\}|}$. There holds row growth 5.2 $H_k(\mathfrak{gl}_n^{\preceq}) \cong H_k(\mathfrak{gl}_{n-1}^{\preceq}) \oplus \dots$ and $H^k(\mathfrak{gl}_n^{\preceq}) \cong H^k(\mathfrak{gl}_{n-1}^{\preceq}) \times \dots$. For every interval in $([n], \preceq)$ with $t+1$ elements, 5.7 $H_{2t-3}(\mathfrak{gl}_n^{\preceq}; \mathbb{Z})$ has a direct summand \mathbb{Z}_t .

If \preceq is the usual total order on $[n]$, so that $\mathfrak{gl}_n^{\preceq} = \mathfrak{sol}_n$ is the solvable Lie algebra of all upper triangular matrices with rank $N = \binom{n+1}{2}$, then 3.10 p -torsion in the homology table 3 appears for the first time in $p+1$ -th column and $2p-1$ -th row. Homological dimension 3.8 of \mathfrak{sol}_n over \mathbb{Q} is n , over \mathbb{Z}_2 is $\begin{cases} N \\ N-n/2 \end{cases}$ if n is odd, over \mathbb{Z}_3 is $N-m$ when $n=3m \pm 1$, and over \mathbb{Z}_p with $p \geq 5$ is smaller than over $N-n/2$.

There holds 6.1 $C_*(\mathfrak{gl}_n^{\preceq}; \mathbb{Z}_p) \simeq C_{*,p}(\mathfrak{gl}_n^{\preceq}; \mathbb{Z}_p) \otimes C_*(\mathfrak{dgn}_n; \mathbb{Z}_p)$ where \mathfrak{dgn}_n is the Lie algebra of diagonals, thus $\dim H_k(\mathfrak{gl}_n^{\preceq}; \mathbb{Z}_p) = \sum_{i+j=k} \dim H_i(C_{*,p}(\mathfrak{gl}_n^{\preceq}; \mathbb{Z}_p)) \cdot \binom{n}{j}$, but also $\{p\text{-torsion of } C_*(\mathfrak{gl}_n^{\preceq}; \mathbb{Z})\} = \{\text{torsion of } C_{*,p}(\mathfrak{gl}_n^{\preceq}; \mathbb{Z})\}$, where $C_{*,p}$ is the chain subcomplex, spanned by all wedges $\dots e_{ij} \dots$ in which for every index i the number of its left appearances minus the number of its right appearances is a multiple of p .

Conventions. Throughout this article, R will denote a commutative unital ring. Any R -module is assumed to be unital. Letter p will always denote a prime number. Notation $R' \leq R$ means that R' is a unital subring of R , e.g. $\mathbb{Z}_p \leq R$ means that R has characteristic p , and $\mathbb{Q} \leq R$ means all integer multiples of 1_R are invertible. Also R^\times is the set of units of R (elements that have a multiplicative inverse). Moreover, $R^\times \cap \mathbb{Z}$ is the set of invertible integer multiples of 1_R . The free R -module on a set I is denoted by $R^{(I)}$. In a digraph (=directed graph), given an edge $v \rightarrow w$, the vertex v is *initial* and w is *terminal*. We denote $[n] = \{1, 2, \dots, n\}$ and $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$.

1. ALGEBRAIC MORSE THEORY

In this section, we recall the main theorem of AMT in a user-friendly form that is convenient for later reference. The main idea is as follows. Given a chain complex (C_*, ∂_*) of free modules, construct a directed graph Γ_{C_*} (vertices are the basis elements of all C_n , edges are the nonzero maps ∂_n), then intelligently choose a subset \mathcal{M} of the edges of Γ_{C_*} which is a matching with nice properties. AMT states that every such edge in \mathcal{M} kills two modules R (which do not contribute to homology), but then the boundary ∂_* may change. Later, we will see that for \mathfrak{nil}_n the boundary indeed changes, but for \mathfrak{sol}_n it happily stays the same.

1.1. Formulation. Suppose we are given a chain complex of R -modules

$$C_* = (C_k, \partial_k)_{k \in \mathbb{Z}} : \dots \longrightarrow \bigoplus_{v \in I_{k+1}} C_{k+1,v} \xrightarrow{\partial_{k+1}} \bigoplus_{v \in I_k} C_{k,v} \xrightarrow{\partial_k} \bigoplus_{v \in I_{k-1}} C_{k-1,v} \longrightarrow \dots,$$

i.e. index sets I_k are arbitrary, $C_{k,v}$ are any modules and ∂_k are any homomorphisms with the property $\partial_k \partial_{k+1} = 0$ for all $k \in \mathbb{Z}$. For $u \in I_k$ and $v \in I_{k-1}$, let $\partial_{k,u,v} = \partial_{uv}$ denote the R -module morphism $C_{k,u} \xrightarrow{\iota} C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\pi} C_{k-1,v}$, where ι is the inclusion and π the projection. The associated *digraph* of C_* , denoted by Γ_{C_*} , is a directed simple graph with vertex set the disjoint union $\bigsqcup_{k \in \mathbb{Z}} I_k$ and edge set $\{(u, v); \exists k \in \mathbb{Z}: u \in I_k, v \in I_{k-1}, \partial_{k,u,v} \neq 0\}$. We denote each edge (u, v) by $u \rightarrow v$.

Let \mathcal{M} be a subset of the set of edges of Γ_{C_*} . The associated *digraph of \mathcal{M}* , denoted $\Gamma_{C_*}^{\mathcal{M}}$, is the digraph Γ_{C_*} with all the edges in \mathcal{M} having reversed orientation,

$$(\{\text{vertices of } \Gamma_{C_*}\}, \{\text{edges of } \Gamma_{C_*} \text{ not in } \mathcal{M}\} \cup \{v \rightarrow u; u \rightarrow v \in \mathcal{M}\}).$$

We say that \mathcal{M} is a *Morse matching* or *acyclic matching* or *gradient vector field* on Γ_{C_*} when the following conditions hold:

- (1) \mathcal{M} is a *matching*, i.e. edges in \mathcal{M} have no common endpoints, i.e. whenever $u \rightarrow v, u' \rightarrow v' \in \mathcal{M}$ then $\{u, v\} \cap \{u', v'\} = \emptyset$;
- (2) for every edge $u \rightarrow v$ in \mathcal{M} , the corresponding map ∂_{uv} is an isomorphism;
- (3) $\Gamma_{C_*}^{\mathcal{M}}$ contains no directed cycles, and for every k there does not exist an infinite path $u_1 \rightarrow v_1 \rightarrow u_2 \rightarrow v_2 \rightarrow \dots$ in $\Gamma_{C_*}^{\mathcal{M}}$ with all $u_1, u_2, \dots \in I_k$.

A vertex in $\Gamma_{C_*}^{\mathcal{M}}$ that is not incident to an edge in \mathcal{M} is called *\mathcal{M} -critical*. We denote $\dot{\mathcal{M}} = \{\mathcal{M}\text{-critical vertices of } \Gamma_{C_*}^{\mathcal{M}}\}$, $\dot{\mathcal{M}}_k = \dot{\mathcal{M}} \cap I_k$, $\dot{C}_k = \bigoplus_{v \in \dot{\mathcal{M}}_k} C_{k,v}$.

If $u \in \dot{\mathcal{M}}_k$ and $v \in \dot{\mathcal{M}}_{k-1}$, then the set of all directed trails in $\Gamma_{C_*}^{\mathcal{M}}$ from u to v with vertices in $I_k \cup I_{k-1}$ is denoted by $\Gamma_{u,v}^{\mathcal{M}}$. By (1) and (3) such ‘zig-zag’ trails are finite paths. If $\gamma = (u = v_1 \rightarrow \dots \rightarrow v_{2i} = v) \in \Gamma_{u,v}^{\mathcal{M}}$, then the *gradient path* along γ is the signed morphism that maps from $C_{k,u}$ to $C_{k-1,v}$ along the path γ , i.e.

$$\partial_\gamma = (-1)^{i-1} \partial_{v_{2i-1}v_{2i}} \partial_{v_{2i-2}v_{2i-1}}^{-1} \dots \partial_{v_5v_4}^{-1} \partial_{v_3v_4} \partial_{v_3v_2}^{-1} \partial_{v_1v_2}.$$

The inverses exist by (2). Notice: $i-1$ is the number of edges of γ that are contained in \mathcal{M} . Define $\dot{\partial}_k: \dot{C}_k \rightarrow \dot{C}_{k-1}$, for $x \in C_{k,u}$ with $u \in \dot{\mathcal{M}}_k$, by the rule

$$\dot{\partial}_k(x) = \sum_{v \in \dot{\mathcal{M}}_{k-1}, \gamma \in \Gamma_{u,v}^{\mathcal{M}}} \partial_\gamma(x).$$

Theorem 1.1 ([4],[8]). *If \mathcal{M} is a Morse matching, then chain complex $(C_k, \partial_k)_{k \in \mathbb{Z}}$ is homotopy equivalent to the chain complex $(\dot{C}_k, \dot{\partial}_k)_{k \in \mathbb{Z}}$.*

Remarks 1.2. • Any homotopy equivalence between chain complexes induces isomorphisms on corresponding homology modules.

• In most cases of usage, all R -modules $C_{k,u}$ will be just R , so that morphism ∂_{uv} is $R \xrightarrow{\cdot r} R$ for some $r \in R$, called the *weight* of edge $u \rightarrow v$. Then ∂_{uv} is an isomorphism iff r is a unit of R , with inverse $R \xleftarrow{\cdot r^{-1}} R$.

• If $\dot{\mathcal{M}}_{k+1} = \emptyset = \dot{\mathcal{M}}_{k-1}$, so that there are no zig-zag paths (hence $\dot{\partial}_{k+1} = 0 = \dot{\partial}_k$), then it just remains to calculate $\dot{\mathcal{M}}_k$ which gives $H_k \cong \bigoplus_{v \in \dot{\mathcal{M}}_k} C_{k,v}$.

• Empty set $\mathcal{M} = \{\}$ is always a (useless) Morse matching, with $\dot{C}_* = C_*$ and $\dot{\partial}_* = \partial_*$.

• The above construction is done for chain complexes, but cochain complexes are formally the same (via $C^k = C_{-k}$ and $\delta^k = \partial_{-k}$), so the AMT theorem can (in principle) also be used for computing cohomology.

• Suppose that each module C_k is free of finite rank, so that the dual of matrix $R^{|I_k|} \xrightarrow{\partial_k} R^{|I_{k-1}|}$ is the transposed matrix $R^{|I_k|} \xleftarrow{\partial_k^t} R^{|I_{k-1}|}$. Then the digraph of $C^* = \text{Hom}_R(C_*, R)$ is the digraph of C_* with all edges reversed, i.e. $\Gamma_{C^*} = \Gamma_{C_*}^{\text{op}}$. If \mathcal{M} is a Morse matching on C_* , then \mathcal{M}^{op} is a Morse matching on C^* , and zig-zag paths in $\Gamma_{C^*}^{\mathcal{M}^{\text{op}}}$ are reversed zig-zag paths in $\Gamma_{C_*}^{\mathcal{M}}$, hence $(\dot{C}^*, \dot{\delta}^*) = (\dot{C}_*, \dot{\partial}_*)^{\text{op}}$. Alternatively (when C_* is any chain R -complex), $C_* \simeq \dot{C}_*$ implies $\text{Hom}_R(C_*, R) \simeq \text{Hom}_R(\dot{C}_*, R)$, so cohomology can be computed directly by dualizing \dot{C}_* .

• For every edge $u \rightarrow v$ that we add to \mathcal{M} , we kill modules $C_{k,u}$ and $C_{k-1,v}$ without

changing the homology of the complex (provided $\mathcal{M} \cup \{u \rightarrow v\}$ is still a Morse matching). Therefore it is usually desirable for \mathcal{M} to be as large as possible.

- On the other hand, the more edges that \mathcal{M} contains, the more zig-zag paths there are in $\Gamma_C^{\mathcal{M}}$, meaning that the sum ∂_k is large. Thus it may also be desirable that \mathcal{M} does not contain too many edges. Finding a good balance is important.

- Notice: Given matchings $\mathcal{M} \subseteq \mathcal{M}'$ in Γ , if digraph $\Gamma^{\mathcal{M}}$ contains a cycle, then so too does $\Gamma^{\mathcal{M}'}$. Thus enlarging a no-good matching is pointless.

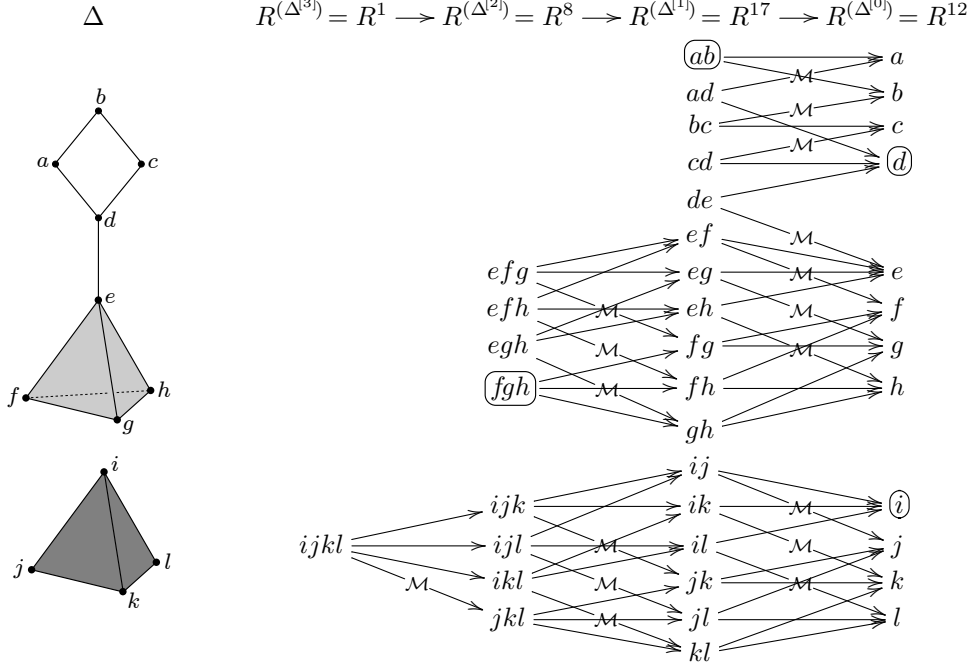
1.2. Examples. Before proceeding to more serious examples, we look at some simple cases to demonstrate how to use AMT. Simplicial and Lie algebra (co)homology behave similarly (vertices of Γ_{C_\star} behave as sets), which is in sharp contrast to group and associative algebra homology (there, vertices behave as sequences, which makes defining \mathcal{M} very different). Therefore, we use the good ol' DMT as our first steps.

Let Δ be a finite-dimensional simplicial complex and $\Delta^{[k]} = \{k\text{-simplices of } \Delta\}$. The (co)homology of Δ is the (co)homology of the Poincaré chain complex

$$\begin{aligned} \dots \longrightarrow R^{\Delta^{[k]}} \longrightarrow R^{\Delta^{[k-1]}} \longrightarrow \dots \longrightarrow R^{\Delta^{[1]}} \longrightarrow R^{\Delta^{[0]}}, \\ \partial\{v_0, \dots, v_k\} = \sum_{i=0}^k (-1)^i \{v_0, \dots, \widehat{v}_i, \dots, v_k\}. \end{aligned}$$

The vertices of Γ_{C_\star} are the simplices of Δ , and for each vertex the corresponding module is isomorphic to R . For a k -simplex σ and $(k-1)$ -simplex τ , the corresponding boundary operator is either zero or an isomorphism: $\partial_{\sigma, \tau} = \begin{cases} \pm \text{id}; & \text{if } \tau \subset \sigma \\ 0 & \text{otherwise} \end{cases}$.

Example 1.3. Consider the 3-dimensional simplicial complex Δ below left. Its simplicial chain complex is depicted below right, together with a chosen Morse matching \mathcal{M} (clearly (1) and (2) hold, but acyclicity (3) must be checked manually). The circled vertices are the \mathcal{M} -critical ones.



Among critical vertices, there are only two zig-zag paths, namely $\gamma: ab \xrightarrow{1} a \xleftarrow{1} ad \xrightarrow{1} d$ and $\gamma': ab \xrightarrow{1} b \xleftarrow{1} bc \xrightarrow{1} c \xleftarrow{1} cd \xrightarrow{1} d$, hence there holds $\partial_\gamma = (-1)^1(-1)(-1)1 = -1$ and $\partial_{\gamma'} = (-1)^2 1(-1)1(-1)1 = 1$, so these two paths cancel each other.

By AMT, this chain complex is homotopy equivalent to $0 \xrightarrow{0} R \xrightarrow{0} R \xrightarrow{(\text{id}-\text{id}, 0)} R^2$, hence the simplicial (co)homology of Δ is

$$H^3 \cong H_3 \cong 0, \quad H^2 \cong H_2 \cong R, \quad H^1 \cong H_1 \cong R, \quad H^0 \cong H_0 \cong R^2. \quad \diamond$$

Example 1.4. The following examples are useful exercises, with which the unfamiliar reader should get his hands dirty, to get more accustomed to the combinatorics of AMT. Drawing pictures of particular cases are highly recommended, as well as checking that M is indeed a Morse matching, and calculating $\dot{\partial}$ is obligatory.

- If $\Delta = \mathbb{B}^n$ is a simplicial n -ball and $\mathcal{M} = \{\sigma \cup \{0\} \rightarrow \sigma; 0 \notin \sigma\}$, then $\dot{\mathcal{M}} = \{\{0\}\}$ and $\dot{\partial} = 0$. Therefore $H_k(\Delta) \cong H^k(\Delta) \cong \begin{cases} R; & k=0 \\ 0; & \text{otherwise} \end{cases}$.
- If $\Delta = \mathbb{S}^n$ is a simplicial n -sphere and $\mathcal{M} = \{\sigma \cup \{0\} \rightarrow \sigma; 0 \notin \sigma\}$, then $\dot{\mathcal{M}} = \{\{0\}, \{1, \dots, n+1\}\}$ and $\dot{\partial} = 0$. Therefore $H_k(\Delta) \cong H^k(\Delta) \cong \begin{cases} R; & k=0, n \\ 0; & \text{otherwise} \end{cases}$.
- If $\Delta = \mathbb{B}_n^1$ is an n -path and $\mathcal{M} = \{\{i-1, i\} \rightarrow \{i\}; i = 1, \dots, n\}$, then $\dot{\mathcal{M}} = \{\{0\}\}$ and $\dot{\partial} = 0$. Therefore $H_k(\Delta) \cong H^k(\Delta) \cong \begin{cases} R; & k=0 \\ 0; & \text{otherwise} \end{cases}$.
- If $\Delta = \mathbb{S}_n^1$ is an n -cycle and $\mathcal{M} = \{\{i-1, i\} \rightarrow \{i\}; i = 2, \dots, n\}$, then $\dot{\mathcal{M}} = \{\{1\}, \{1, n\}\}$ and $\dot{\partial} = 0$. Therefore $H_k(\Delta) \cong H^k(\Delta) \cong \begin{cases} R; & k=0, 1 \\ 0; & \text{otherwise} \end{cases}$.
- Let $\Delta = \bigvee_{i \in I} \Delta_i$ be a wedge sum of arbitrarily many simplicial complexes, joined at 0-simplex v (so that $\Delta^{[k]} = \bigcup_i \Delta_i^{[k]}$ for $k \geq 1$). In the digraph for the chain complex giving reduced homology of Δ (just one additional vertex \emptyset in degree -1 that connects to all 0-simplices), define $\mathcal{M} = \{\{v\} \rightarrow \emptyset\}$. In the digraph for the direct sum of chain complexes giving reduced homology of each Δ_i ($|I|$ more copies of the vertex \emptyset in degree -1 that connects to all 0-simplices), define $\mathcal{M} = \{\{v_i\} \rightarrow \emptyset_i; i \in I\}$. In both cases, the newly obtained complex is the same. This shows that

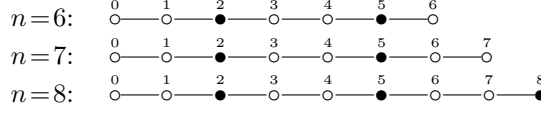
$$\tilde{H}_k(\Delta) \cong \bigoplus_i \tilde{H}_k(\Delta_i) \quad \text{and} \quad \tilde{H}^k(\Delta) \cong \prod_i \tilde{H}^k(\Delta_i).$$

- Let $\Delta = \Delta^{[0]} \cup \Delta^{[1]} = \binom{[n]}{k} \cup \{\{u, v\}; u \cap v = \emptyset\}$ be the (n, k) -th Kneser graph (which is a 1-dimensional simplicial complex, vertices are the k -element subsets of $[n]$, edges correspond to pairs of subsets with empty intersection). Denote $m = \binom{n}{k}$. If $2k > n$, then $\Delta = \bigcup_{i=1}^m \text{pt}$. If $2k = n$, then $\Delta \simeq \bigcup_{i=1}^{m/2} \text{pt}$. But if $2k < n$, define $\varphi: \binom{[n]}{k} \rightarrow \binom{[n]}{k}$

$$\varphi: u \mapsto \begin{cases} \text{first } k \text{ elements of } [n] \setminus u & ; u \not\subseteq [2k] \\ [2k] \setminus u \text{ without minimum but with } 2k+1; & u \subseteq [2k] \end{cases}$$

and let $\mathcal{M} = \{\{u, \varphi(u)\} \rightarrow \{u\}; u \neq [k]\}$. Then $\forall u \exists i: \varphi^i(u) = [k]$ (hence there are no cycles in $\Gamma_{C_*}^{\mathcal{M}}$) and $\dot{\mathcal{M}} = \{\{[k]\}, \{u, v\}; \varphi(u) \neq v\}$ and finally $\Delta \simeq \bigvee_{i=1}^N \mathbb{S}^1$ where $N = \binom{n}{k} \binom{n-k}{k} / 2 - \binom{n}{k} + 1$. Recall that any connected graph with v vertices and e edges is homotopy equivalent to a wedge of $e - v + 1$ copies of \mathbb{S}^1 (contract a maximal tree). But here we obtain an explicit spanning tree in Δ , namely $\{\{u, \varphi(u)\}; u \neq [k]\}$.

- Let \mathbb{B}_n^1 be an n -path (vertices $0, 1, \dots, n$, edges $\{0, 1\}, \{1, 2\}, \dots, \{n-1, n\}$). Let Δ be the anticlique complex of \mathbb{B}_n^1 (vertices of Δ are the vertices of \mathbb{B}_n^1 , simplices of Δ are the sets of vertices of \mathbb{B}_n^1 which contain no edges). Define a matching by $\mathcal{M} = \{\sigma \rightarrow \sigma \setminus \{3i\}; \min \sigma \cap (3\mathbb{N}+2) = 3i+2, 3i \in \sigma\}$ so that $\dot{\mathcal{M}} = \{\sigma; \sigma \cap (3\mathbb{N}+2) = \emptyset\}$.



Let $n = 3k + r$ with $r \in \{0, 1, 2\}$. If $r = 0$ and we add $\{\sigma \rightarrow \sigma \setminus \{n\}; n \in \sigma\}$ to \mathcal{M} , then $\mathring{\mathcal{M}} = \{\{n\}\}$. If $r \neq 0$, then the critical simplices of Δ form the simplicial join $\mathring{\mathcal{M}} \approx \mathring{\mathbb{S}}^0 \approx \mathbb{S}^k$, so adding edges $\{\sigma \rightarrow \sigma \setminus \{m+1\}; m = \min 3\mathbb{N} \setminus \sigma, m+1 \in \sigma\}$ to \mathcal{M} gives $\mathring{\mathcal{M}} = \{\{1\}, \{0, 3, 6, \dots, 3k\}\}$. We conclude that $\Delta \simeq \begin{cases} \text{pt}; & n=3k \\ \mathbb{S}^k; & n=3k+1, 2 \end{cases}$. \diamond

2. CHEVALLEY (Co)HOMOLOGY

In this section we recall the definitions and examples of Lie algebras and their (co)homology, from the viewpoint of digraphs and AMT.

2.1. Formulation. [11, 7] Let \mathfrak{g} be a Lie R -algebra, i.e. an R -module with a binary operation $[-, -]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ that is R -linear in both variables and satisfies identities

$$\forall a, b, c \in \mathfrak{g}: [a, a] = 0, [[a, b], c] + [[b, c], a] + [[c, a], b] = 0.$$

Let $\Lambda^k M = (\bigotimes_{i=1}^k M) / \langle m_1 \otimes \dots \otimes m_k; m_i = m_j \text{ for some } i \neq j \rangle$ be the exterior power of R -module M . The *Chevalley (co)homology* of \mathfrak{g} with trivial coefficients, denoted $H_k(\mathfrak{g}; R)$ and $H^k(\mathfrak{g}; R)$, is the (co)homology of the chain complex of R -modules

$$\begin{aligned}
C_\star: \quad & \dots \rightarrow \Lambda^k \mathfrak{g} \xrightarrow{\partial_k} \Lambda^{k-1} \mathfrak{g} \rightarrow \dots \rightarrow \Lambda^2 \mathfrak{g} \xrightarrow{[\cdot, \cdot]} \mathfrak{g} \xrightarrow{0} R, \\
\partial_k(x_1 \wedge \dots \wedge x_k) = & \sum_{r < s} (-1)^{r+s} [x_r, x_s] \wedge x_1 \wedge \dots \wedge \widehat{x_r} \wedge \dots \wedge \widehat{x_s} \wedge \dots \wedge x_k.
\end{aligned}$$

Thus $H_0(\mathfrak{g}; R) \cong H^0(\mathfrak{g}; R) \cong R$. We shall deal only with Lie algebras that admit some R -module basis $\{e_i; i \in I\} \subseteq \mathfrak{g}$, because then any linear ordering \leq of I implies that $\{e_{i_1} \wedge \dots \wedge e_{i_k}; i_1 < \dots < i_k\}$ is a basis of $\Lambda^k \mathfrak{g}$, so $\{e_{i_1} \wedge \dots \wedge e_{i_k}; i_1 < \dots < i_k, k \in \mathbb{N}\}$ is the set of vertices of Γ_{C_\star} . From now on, we will omit the \wedge symbols to save space, and remember that $e_{\pi_{i_1}} \dots e_{\pi_{i_k}} = \text{sgn } \pi e_{i_1} \dots e_{i_k}$ for any permutation π .

If we denote $H^*(\mathfrak{g}) = \bigoplus_{k \in \mathbb{N}} H^k(\mathfrak{g})$, then diagonal map $\mathfrak{g} \rightarrow \mathfrak{g} \times \mathfrak{g}, x \mapsto (x, x)$ and zero map $\mathfrak{g} \rightarrow 0, x \mapsto 0$ induce *cup product* $\smile: H^*(\mathfrak{g}) \otimes H^*(\mathfrak{g}) = H^*(\mathfrak{g} \times \mathfrak{g}) \rightarrow H^*(\mathfrak{g})$ and $R = H^*(0) \rightarrow H^*(\mathfrak{g})$ that make $H^*(\mathfrak{g})$ an associative unital graded-commutative R -algebra. Explicitly, for $\alpha \in H^i(\mathfrak{g})$ and $\beta \in H^j(\mathfrak{g})$ we have

$$(\alpha \smile \beta)(x_1 \wedge \dots \wedge x_{i+j}) = \sum_{\substack{\pi \in S_{i+j}, \pi_1 < \dots < \pi_i, \\ \pi_{i+1} < \dots < \pi_{i+j}}} \text{sgn } \pi \alpha(x_{\pi_1} \wedge \dots \wedge x_{\pi_i}) \beta(x_{\pi_{i+1}} \wedge \dots \wedge x_{\pi_{i+j}}),$$

where the sum runs through all shuffle permutations.

2.2. Operations on complexes. If R is a PID (such as \mathbb{Z}) and \mathfrak{g} as an R -module is free of finite rank N , then $\Lambda^k \mathfrak{g}$ is free of rank $\binom{N}{k}$ and $H_k(\mathfrak{g}), H^k(\mathfrak{g})$ are finitely generated R -modules, subject to the classification theorem. Denote $F(-)$ = free part, $T(-)$ = torsion part, $T_p(-)$ = p -torsion part. Universal Coefficient Theorem states: $F(H^k(\mathfrak{g})) \cong F(H_k(\mathfrak{g}))$, $T(H^k(\mathfrak{g})) \cong T(H_{k-1}(\mathfrak{g}))$; $H_k(\mathfrak{g}; \mathbb{Z}_p) \cong (F(H_k(\mathfrak{g}; \mathbb{Z}) \oplus T_p(H_k(\mathfrak{g}; \mathbb{Z}) \oplus T_p(H_{k-1}(\mathfrak{g}; \mathbb{Z})) \otimes \mathbb{Z}_p$. Künneth Theorem states: if R is a hereditary

ring and C_\star and D_\star are chain complexes of R -modules, then

$$H_k(\text{Hom}(C_\star, D_\star)) \cong \prod_{i+j=k} \text{Hom}(H_i(C_\star), H_j(D_\star)) \times \prod_{i+j=k-1} \text{Ext}^1(H_i(C_\star), H_j(D_\star))$$

when all C_i are projective or all D_j are injective, and

$$H_k(C_\star \otimes D_\star) \cong \bigoplus_{i+j=k} H_i(C_\star) \otimes H_j(D_\star) \oplus \bigoplus_{i+j=k-1} \text{Tor}_1(H_i(C_\star), H_j(D_\star))$$

when all C_i or all D_j are flat.

2.3. Examples. For any associative R -algebra A we obtain a Lie R -algebra $L(A)$: as an R -module it is the same as A , but the bracket is given by $[a, b] = ab - ba$. Let $M_n(R)$ denote the associative unital R -algebra of all $n \times n$ matrices and $\mathfrak{gl}_n(R) = L(M_n(R))$. We have subalgebras $M_n^{\geq}(R) = \{a \in M_n(R); a_{ij} = 0 \text{ for } i \not\leq j\}$ and $M_n^{\leq}(R) = \{a \in M_n(R); a_{ij} = 0 \text{ for } i \not\leq j\}$ of all (strictly) upper triangular matrices, together with their Lie counterparts $\mathfrak{sol}_n(R) = L(M_n^{\geq}(R))$ and $\mathfrak{nil}_n(R) = L(M_n^{\leq}(R))$. More generally, let (P, \preceq) be any finite poset (or equivalently, an acyclic digraph) with elements p_1, \dots, p_n . We obtain associative R -algebras and Lie R -algebras

$$\begin{aligned} M_n^{\preceq}(R) &= \{a \in M_n(R); a_{ij} = 0 \text{ for } p_i \not\preceq p_j\}, & \mathfrak{gl}_n^{\preceq}(R) &= L(M_n^{\preceq}(R)), \\ M_n^{\succ}(R) &= \{a \in M_n(R); a_{ij} = 0 \text{ for } p_i \not\succ p_j\}, & \mathfrak{gl}_n^{\succ}(R) &= L(M_n^{\succ}(R)). \end{aligned}$$

If \preceq is the usual linear ordering of $[n]$, then $\mathfrak{gl}_n^{\preceq}(R) = \mathfrak{sol}_n(R)$ and $\mathfrak{gl}_n^{\succ}(R) = \mathfrak{nil}_n(R)$. For theory it is convenient to assume $p_i = i$, but for interesting examples one needs abstract p_i (such as when our poset is $(2^{[n]}, \subseteq)$).

2.4. Motivation. Traditionally, the coefficients of most interest for (co)homology of a given Lie algebra are fields of characteristic 0, because they calculate the singular (co)homology of the corresponding Lie group. Lie groups and their (co)homology are important in differential geometry/topology, because they are the objects that act on smooth manifolds, i.e. the appropriate representations of automorphism groups in the category of smooth manifolds. Smooth manifolds are analytic constructs, their algebraic analogue being *algebraic sets* (solutions of a system of polynomial equations in several variables), which are defined over any field K , and the analogue of Lie groups are the *algebraic groups* (algebraic sets with group structure, such that multiplication and inversion are polynomial maps). We expect a similar correspondence between algebraic groups over K and Lie K -algebras; see [10]. Thus along with $H_k(\mathfrak{g}; \mathbb{Q})$, it is also desirable to know about $H_k(\mathfrak{g}; \mathbb{Z}_p)$ for all p .

3. TRIANGULAR MATRICES $\mathfrak{sol}_n(R)$

Our Lie algebra $\mathfrak{sol}_n(R)$ is a subalgebra of $\mathfrak{gl}_n(R)$ and is solvable. It admits an R -module basis $\{e_{ij}; 1 \leq i \leq j \leq n\}$ where e_{ij} is the matrix with 1 at (i, j) -th entry and 0 elsewhere. Programming with large matrices shows $H_k(\mathfrak{sol}_n; \mathbb{Z})$ for small n below. This is an infinite table of \mathbb{Z} -modules. In this section, C_\star denotes the chain complex for the homology of $\mathfrak{g} = \mathfrak{sol}_n$. Since \mathfrak{g} as an R -module is free of rank $N = \binom{n+1}{2}$, the matrix ∂_k is of size $\binom{N}{k-1} \times \binom{N}{k}$ and Γ_{C_\star} has 2^N vertices. This exponential growth is problematic for computing the Smith normal form of a matrix: the largest matrices appear at the middle of the complex and are of width

$$\binom{N}{N/2} \left| \begin{array}{cccccccccccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\ 10^1 & 10^1 & 10^2 & 10^3 & 10^4 & 10^6 & 10^8 & 10^{10} & 10^{13} & 10^{16} & 10^{19} & 10^{23} & 10^{27} & 10^{31} & 10^{35} & 10^{40} & 10^{45} & 10^{51} & 10^{56} & 10^{62} \end{array} \right|$$

Dealing with such size is not feasible, even though the matrices are very sparse. For reference, the number of all atoms of the Earth is $\doteq 10^{50}$.

$k \backslash n$	1	2	3	4	5	6
0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}
1	\mathbb{Z}	\mathbb{Z}^2	\mathbb{Z}^3	\mathbb{Z}^4	\mathbb{Z}^5	\mathbb{Z}^6
2		\mathbb{Z}	\mathbb{Z}^3	\mathbb{Z}^6	\mathbb{Z}^{10}	\mathbb{Z}^{15}
3		0	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}^4 \oplus \mathbb{Z}_2^3$	$\mathbb{Z}^{10} \oplus \mathbb{Z}_2^6$	$\mathbb{Z}^{20} \oplus \mathbb{Z}_2^{10}$
4			\mathbb{Z}_2^2	$\mathbb{Z} \oplus \mathbb{Z}_2^{11}$	$\mathbb{Z}^5 \oplus \mathbb{Z}_2^{29}$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^{59}$
5			\mathbb{Z}_2	$\mathbb{Z}_2^{15} \oplus \mathbb{Z}_3$	$\mathbb{Z} \oplus \mathbb{Z}_2^{56} \oplus \mathbb{Z}_3^3$	$\mathbb{Z}^6 \oplus \mathbb{Z}_2^{145} \oplus \mathbb{Z}_3^6$
6			0	$\mathbb{Z}_2^9 \oplus \mathbb{Z}_3^3$	$\mathbb{Z}_2^{59} \oplus \mathbb{Z}_3^{13}$	$\mathbb{Z} \oplus \mathbb{Z}_2^{220} \oplus \mathbb{Z}_3^{33}$
7				$\mathbb{Z}_2^2 \oplus \mathbb{Z}_3^3$	$\mathbb{Z}_2^{51} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3^{22}$	$\mathbb{Z}_2^{348} \oplus \mathbb{Z}_4^3 \oplus \mathbb{Z}_3^{75}$
8				\mathbb{Z}_3	$\mathbb{Z}_2^{55} \oplus \mathbb{Z}_4^4 \oplus \mathbb{Z}_3^{19}$	$\mathbb{Z}_2^{674} \oplus \mathbb{Z}_4^{16} \oplus \mathbb{Z}_3^{96}$
9				0	$\mathbb{Z}_2^{50} \oplus \mathbb{Z}_4^6 \oplus \mathbb{Z}_3^{11}$	$\mathbb{Z}_2^{1034} \oplus \mathbb{Z}_4^{35} \oplus \mathbb{Z}_3^{94} \oplus \mathbb{Z}_5^1$
10				0	$\mathbb{Z}_2^{26} \oplus \mathbb{Z}_4^4 \oplus \mathbb{Z}_3^7$	$\mathbb{Z}_2^{1035} \oplus \mathbb{Z}_4^{40} \oplus \mathbb{Z}_3^{101} \oplus \mathbb{Z}_5^5$
11					$\mathbb{Z}_2^9 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3^4$	$\mathbb{Z}_2^{704} \oplus \mathbb{Z}_4^{25} \oplus \mathbb{Z}_3^{103} \oplus \mathbb{Z}_5^{10}$
12					$\mathbb{Z}_2^6 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2^{452} \oplus \mathbb{Z}_4^9 \oplus \mathbb{Z}_3^{70} \oplus \mathbb{Z}_5^{10}$
13					\mathbb{Z}_2^4	$\mathbb{Z}_2^{389} \oplus \mathbb{Z}_4^6 \oplus \mathbb{Z}_3^{26} \oplus \mathbb{Z}_5^5$
14					\mathbb{Z}_2	$\mathbb{Z}_2^{305} \oplus \mathbb{Z}_4^{10} \oplus \mathbb{Z}_3^4 \oplus \mathbb{Z}_5$
15					0	$\mathbb{Z}_2^{150} \oplus \mathbb{Z}_4^{10}$
16						$\mathbb{Z}_2^{39} \oplus \mathbb{Z}_4^5$
17						$\mathbb{Z}_2^4 \oplus \mathbb{Z}_4$
18						0
19						0
20						0
21						0

3.1. Digraph. Every edge of Γ_{C_\star} is of the form $\begin{smallmatrix} \dots e_{aa} e_{ab} \dots \\ \dots e_{ab} \dots \end{smallmatrix}$ or $\begin{smallmatrix} \dots e_{ab} e_{bb} \dots \\ \dots e_{ab} \dots \end{smallmatrix}$ or $\begin{smallmatrix} \dots e_{ax} e_{xb} \dots \\ \dots e_{ab} \dots \end{smallmatrix}$ for some $a < x < b$, because $[e_{ab}, e_{cd}] = \delta_{bc} e_{ad} - \delta_{ad} e_{cb}$. The last edge has weight ± 1 , but this may not hold for the first two. For any $\sigma = \{x_1, \dots, x_k\} \subseteq [n]$ we denote $e_\sigma = e_{x_1 x_1} \dots e_{x_k x_k}$, the wedge of certain diagonals.

Proposition 3.1. $\{\text{isolated vertices of } \Gamma_{C_\star}\} = \{e_\sigma; \sigma \subseteq [n]\}$ when $\mathbb{Z} \leq R$.

Thus $R^{(n)}_k$ is a direct summand of $H_k(\mathfrak{sol}_n)$ and $H^k(\mathfrak{sol}_n)$.

Proof. The Lie bracket of any two diagonals e_{aa} and e_{bb} is 0. The only splitting of a diagonal is $e_{aa} = e_{ax} \cdot e_{xa}$, but then either $e_{ax} \notin \mathfrak{sol}_n$ or $e_{xa} \notin \mathfrak{sol}_n$. Thus every e_σ is isolated. If a vertex v contains a nondiagonal element e_{ab} , then v either contains e_{aa} (hence omitting e_{aa} makes v initial) or it doesn't (hence adding e_{aa} makes v terminal). Thus every isolated vertex is of the form e_σ . Here is the tricky part: adding or removing a diagonal can be an edge with weight 0 (so not really an edge). But every vertex that contains a nondiagonal element also contains a nondiagonal index with nonzero weight; see the proof of 3.9 and 3.5 below. \square

Over any ring R , there holds row growth:

Proposition 3.2. $H_k(\mathfrak{sol}_n) \cong H_k(\mathfrak{sol}_{n-1}) \oplus \dots$ and $H^k(\mathfrak{sol}_n) \cong H^k(\mathfrak{sol}_{n-1}) \times \dots$

Proof. The largest and the smallest index that appears in a vertex v appears also in any vertex adjacent to v . Hence chain complex $C_\star(\mathfrak{sol}_n)$ is a direct sum of $C_\star(\mathfrak{sol}_{n-1})$ and the subcomplex spanned by all vertices that contain index n . \square

Proposition 3.3. $H_{2n-3}(\mathfrak{so}l_n; \mathbb{Z}) \cong \mathbb{Z}_{n-1} \oplus \dots$

Thus every prime power \mathbb{Z}_{p^l} appears in the table as a direct summand.

Proof. The vertex $v = e_{12}e_{13}e_{14}\dots e_{1n}e_{2n}e_{3n}\dots e_{n-1,n}$ is initial to no edge (hence $v \in \text{Ker } \partial_{2n-3}$) and terminal to two edges, namely to $e_{\{1\}}v \rightarrow v$ and $e_{\{n\}}v \rightarrow v$, whose weights are $\pm(n-1)$ (hence $(n-1)v \in \text{Im } \partial_{2n-2}$). \square

Conjecture 3.4. \mathbb{Z}_m with $m=p^l$ appears for the first time in $m+1$ -th column and $2m-1$ -th row.

3.2. Matching. The increasing wedges of e_{ab} 's constitute the bases for modules $C_k(\mathfrak{g})$. By permuting elements of a wedge, we obtain the same wedge together with the sign of the permutation. Such permuted wedges constitute new bases for $C_k(\mathfrak{g})$. Every vertex of digraph Γ_{C_\star} can be written in the form $v = \pm e_{a_1 b_1} \dots e_{a_l b_l} e_\sigma$ with all $a_i < b_i$, and then we denote $\epsilon(v) = \{a_1, b_1, \dots, a_l, b_l\}$. For any index $x \in \epsilon(v)$ we can write vertex v in the form $\pm e_{a_1 x} \dots e_{a_r x} \overline{e_{xx}} e_{x b_1} \dots e_{x b_s} e_{c_1 d_1} \dots e_{c_t d_t} e_\sigma$ with $x \notin \{c_1, d_1, \dots, c_t, d_t\}$ and $x \notin \sigma$, where overline means e_{xx} is either missing or present in the wedge, so omitting e_{xx} is an edge with weight $(-1)^r(r-s)$. Here $r = r_x$ is the number of times x appears on the right and $s = s_x$ is the number of times x appears on the left. We define

$$\mathcal{M} = \left\{ \begin{array}{l} e_{a_1 x} \dots e_{a_r x} \overline{e_{xx}} e_{x b_1} \dots e_{x b_s} e_{c_1 d_1} \dots e_{c_t d_t} e_\sigma, \quad x \in \epsilon(v) \text{ is minimal such} \\ e_{a_1 x} \dots e_{a_r x} \overline{e_{xx}} e_{x b_1} \dots e_{x b_s} e_{c_1 d_1} \dots e_{c_t d_t} e_\sigma, \quad \text{that } r-s \text{ is a unit of } R \end{array} \right\}.$$

Proposition 3.5. \mathcal{M} is a Morse matching, with $\mathring{\partial} = \partial$ and

$$\mathring{\mathcal{M}} = \{v; \forall x \in \epsilon(v): r_x - s_x \in R \setminus R^\times\}.$$

Thus only the invertibility of integer multiples of unity 1_R determine \mathcal{M} .

Proof. Let us make sure that the edge above truly has weight $(-1)^r(r-s)$. Obtaining

$$\begin{array}{c} e_{a_1 x} \dots e_{a_i x} \dots e_{a_r x} \overline{e_{xx}} e_{x b_1} \dots e_{x b_j} \dots e_{x b_s} e_{c_1 d_1} \dots e_{c_t d_t} e_\sigma \\ \downarrow \\ e_{a_1 x} \dots e_{a_i x} \dots e_{a_r x} \overline{e_{xx}} e_{x b_1} \dots e_{x b_j} \dots e_{x b_s} e_{c_1 d_1} \dots e_{c_t d_t} e_\sigma \end{array}$$

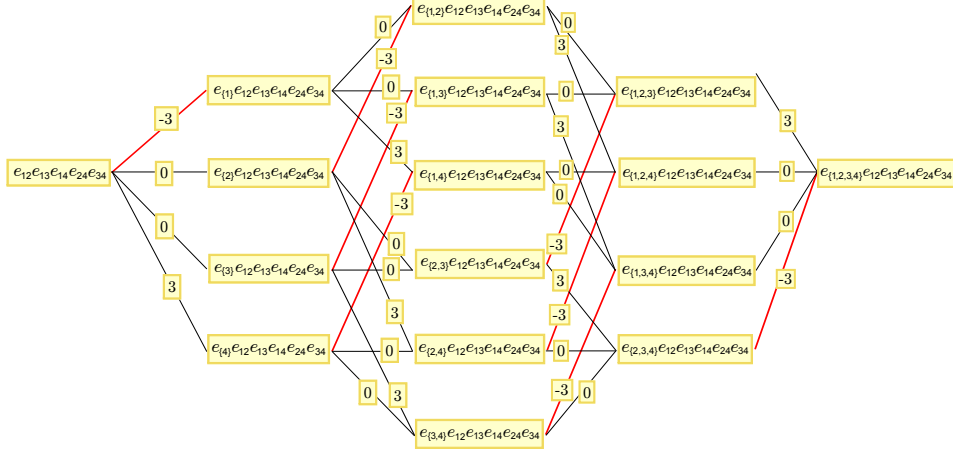
is only possible when we take the bracket $[e_{a_i x}, e_{xx}] = e_{a_i x}$ or $[e_{xx}, e_{x b_j}] = e_{x b_j}$ for all i and j , because for all other nonzero brackets the element e_{xx} remains in the wedge. The first bracket has sign $(-1)^{i+(r+1)+(i-1)} = (-1)^r$ and there are r choices for i . The second bracket has sign $(-1)^{(r+1)+(r+1+j)+(r+j-1)} = -(-1)^r$ and there are s choices for j . Together this gives sign $(-1)^r(r-s)$ and by definition of \mathcal{M} this is a unit of R , so condition (2) from 1.1 is satisfied.

For every edge in \mathcal{M} , the initial and terminal vertex have the same set $\epsilon(-)$, so $\xrightarrow{\mathcal{M}} \xrightarrow{\mathcal{M}}$ does not occur in Γ_{C_\star} , because we cannot twice remove e_{xx} . Since x is uniquely determined by the set $\epsilon(v)$, situations $\xleftarrow{\mathcal{M}} \xrightarrow{\mathcal{M}}$ and $\xrightarrow{\mathcal{M}} \xleftarrow{\mathcal{M}}$ do not occur in Γ_{C_\star} . This shows that condition (1) from 1.1 is satisfied.

Since digraph Γ_{C_\star} is finite, there cannot exist an infinite zig-zag path in $\Gamma_{C_\star}^\mathcal{M}$. For $\begin{smallmatrix} u = \dots e_{ax} e_{xb} \dots \\ v = \dots e_{ab} \dots \end{smallmatrix}$ and any $i \in \epsilon(u)$, number $r_i - s_i$ calculated in u and in v is the same. Given $u \xrightarrow{\mathcal{M}} v$ which is the omission of e_{xx} , for any edge $u' \rightarrow v$, going from v to u' is either adding some $e_{yy} \notin v$ or splitting some $e_{ab} \in v$. In both cases, the minimal $y \in \epsilon(u')$ with $r_y - s_y \in R^\times$ is still x , but since $e_{xx} \notin u'$, the edge $u' \xrightarrow{\mathcal{M}} v$ is not possible and u' is terminal (hence not critical). Thus there are no directed cycles

in $\Gamma_{C_*}^{\mathcal{M}}$, affirming condition (3) from 1.1, and every zig-zag path between critical vertices contains no edge from \mathcal{M} , so ∂ is the restriction of ∂ to \mathcal{M} . \square

Example 3.6. Below is a subgraph corresponding to a direct summand of the chain complex for \mathfrak{so}_4 . The edges are labeled by their weight.



If $\mathbb{Z}_3 \leq R$, then all weights are zero, so there are no edges and every vertex is a generator in the (co)homology. But if $\mathbb{Q} \leq R$ or $\mathbb{Z}_p \leq R$ where $p \neq 3$, then the weights ± 3 are invertible, so the matching \mathcal{M} defined above is marked by red lines: there are no critical vertices, hence the (co)homology is trivial.

3.3. Low Degrees. We are interested in the first three rows of the table.

Proposition 3.7. *If $\mathbb{Z} \cap R^\times = \{\pm 1\}$, such as in the case $R = \mathbb{Z}$, then there holds $H_1(\mathfrak{so}_n) \cong H^1(\mathfrak{so}_n) \cong R^n$, $H_2(\mathfrak{so}_n) \cong H^2(\mathfrak{so}_n) \cong R^{\binom{n}{2}}$, $H_3(\mathfrak{so}_n) \cong R^{\binom{n}{3}} \oplus (\frac{R}{2R})^{\binom{n-1}{2}}$.*

Proof. A nice combinatorial exercise finds the elements of $\mathring{\mathcal{M}}$ and their boundary:

$$\begin{aligned}
 \mathring{\mathcal{M}}_1: \quad & \partial e_{\{a\}} = 0; & a < b < c < d, \quad x \in \{a, b, c, d\}, \quad w_x = r_x - s_x \\
 \mathring{\mathcal{M}}_2: \quad & \partial e_{\{a, b\}} = 0; \\
 \mathring{\mathcal{M}}_3: \quad & \partial e_{\{a, b, c\}} = 0, \partial e_{ab}e_{ac}e_{bc} = 0, \quad w_x = -2, 0, 2; \\
 \mathring{\mathcal{M}}_4: \quad & \partial e_{\{a, b, c, d\}} = 0, \partial e_{\{d\}}e_{ab}e_{ac}e_{bc} = \begin{cases} \pm 2e_{ab}e_{ac}e_{bc}; & \text{if } d \in \{a, c\} \\ 0 & \text{otherwise} \end{cases}, \\
 & \partial e_{ab}e_{ac}e_{bd}e_{cd} = -e_{ac}e_{ad}e_{cd} - e_{ab}e_{ad}e_{bd}, \quad w_x = -2, 0, 0, 2, \\
 & \partial e_{ab}e_{ad}e_{bc}e_{cd} = e_{ac}e_{ad}e_{cd} - e_{ab}e_{ad}e_{bd}, \quad w_x = -2, 0, 0, 2, \\
 & \partial e_{ac}e_{ad}e_{bc}e_{bd} = 0, \quad w_x = -2, -2, 2, 2.
 \end{aligned}$$

Therefore $H_3(\mathfrak{g}) \cong \frac{\text{Ker } \partial_3}{\text{Im } \partial_4} \cong \frac{\langle e_{\{a, b, c\}}, e_{ab}e_{ac}e_{bc} \rangle}{\langle e_{ac}e_{ad}e_{cd} - e_{ab}e_{ad}e_{bd}, 2e_{ab}e_{ac}e_{bc} \rangle}$. Now $e_{\{a, b, c\}}$ is untouched by the relations, so it gives a summand $R^{\binom{n}{3}}$. For the rest, relation $2e_{ab}e_{ac}e_{bc}$ implies we have only $\frac{R}{2R}$ summands, and relation $e_{ac}e_{ad}e_{cd} = e_{ab}e_{ad}e_{bd}$ determines how many: since b is interchangeable with any c , it might as well be just $a+1$, so as many as the cardinality of set $\{(a, d); 1 \leq a < a+1 < d \leq n\}$, which is $\binom{n-1}{2}$. \square

3.4. High Degrees. We study the last few homologies in each column of the table, namely the strange patterns of zeros at the bottom. We wish to know when the first (from the bottom up) nonzero module appears, and over which \mathbb{Z}_p . Let $N = \binom{n+1}{2}$ be the rank of \mathfrak{g} . For any prime p we denote by $T_p(-)$ the p -torsion of a \mathbb{Z} -module.

Proposition 3.8. *If n is odd, $H_N(\mathfrak{sol}_n; \mathbb{Z}) \cong 0 \not\cong H_{N-1}(\mathfrak{sol}_n; \mathbb{Z}) = T_2 H_{N-1}(\mathfrak{sol}_n; \mathbb{Z})$. Let n be even. Then we have $H_{N-k}(\mathfrak{sol}_n; \mathbb{Z}_2) \cong 0 \not\cong H_{N-n/2}(\mathfrak{sol}_n; \mathbb{Z}_2)$ for $k < n/2$ and $H_{N-k}(\mathfrak{sol}_n; \mathbb{Z}_3) \cong 0$ for $k < n/3$. Also $H_{N-m}(\mathfrak{sol}_n; \mathbb{Z}_3) \not\cong 0$ if $n = 3m \pm 1$. If $p \geq 5$, then p -torsion appears later (higher in the column) than 2-torsion or 3-torsion.*

Proof. Denote $e = \bigwedge_{a < b} e_{ab} \in C_N$, the wedge of all elements, and notice that weight is $w_i = r_i - s_i = (i-1) - (n-i) = 2i - n - 1$ for any $i \in [n]$, which is odd iff n is even. Thus $C_N = \langle e \rangle$, $C_{N-1} = \langle e \setminus e_{aa}, e \setminus e_{ab} \rangle$, $C_{N-2} = \langle e \setminus e_{aa}e_{bb}, e \setminus e_{aa}e_{bc}, e \setminus e_{ab}e_{cd} \rangle$, etc. Since $\partial(e) = \sum_x w_x e \setminus e_{xx} \neq 0$, we have $H_N(\mathfrak{g}; \mathbb{Z}) \cong 0$. If n is odd, then over \mathbb{Z}_2 all weights are 0, so $\dot{\mathcal{M}}_N = \{e\}$ and $\partial_N = 0$, and over \mathbb{Z}_p with $p \geq 3$ some weights are not 0, so $\dot{\mathcal{M}}_N = \{\}$. This shows that $H_N(\mathfrak{g}; \mathbb{Z}_2) \cong \mathbb{Z}_2$ and $H_N(\mathfrak{g}; \mathbb{Z}_p) \cong 0$, consequently $H_{N-1}(\mathfrak{g}; \mathbb{Z})$ contains 2-torsion but no p -torsion.

Let n be even. Removing a diagonal from e does not change weights. Removing e_{ab} from e decreases r_b and s_a by 1, hence decreases w_a and increases w_b by 1. For $i = n, \dots, 2, 1$ we have $w_i = n-1, n-3, n-5, \dots, 3, 1, -1, -3, \dots, 5-n, 3-n, 1-n$. Over \mathbb{Z}_2 this is $w_i \equiv 1$ for all i . Thus to obtain a critical vertex (make all weights zero), we must remove at least $n/2$ nondiagonals, so $H_{N-k}(\mathfrak{g}; \mathbb{Z}_2) \cong 0$ for $k < n/2$. Since $\dot{\mathcal{M}}_{N-n/2}$ contains $e \setminus e_{12}e_{34} \cdots e_{n-1,n}$ whose boundary is 0, we get $H_{N-n/2}(\mathfrak{g}; \mathbb{Z}_2) \not\cong 0$.

Over \mathbb{Z}_3 for $n \in 2\mathbb{N}$ there are three cases in which the weights appear:

$$\begin{aligned} n = 3m-1: & \quad w_i \equiv 1, -1, 0, 1, -1, 0, 1, -1, 0, \dots, 1, -1, \\ n = 3m: & \quad w_i \equiv -1, 0, 1, -1, 0, 1, -1, 0, 1, \dots, -1, 0, 1, \\ n = 3m+1: & \quad w_i \equiv 0, 1, -1, 0, 1, -1, 0, 1, -1, \dots, 0, 1, -1, 0. \end{aligned}$$

In all cases, we must remove at least m nondiagonals to obtain a critical vertex, so $H_{N-k}(\mathfrak{g}; \mathbb{Z}_3) \cong 0$ for $k < m$. In the first case, $e \setminus e_{n,n-1}e_{n-3,n-4}e_{n-6,n-7} \cdots e_{2,1} \in \text{Ker } \partial$, hence $H_{N-m}(\mathfrak{g}; \mathbb{Z}_3) \not\cong 0$. In the third case, $e \setminus e_{n-1,n-2}e_{n-4,n-5}e_{n-7,n-8} \cdots e_{3,2} \in \text{Ker } \partial$, hence $H_{N-m}(\mathfrak{g}; \mathbb{Z}_3) \not\cong 0$. The second case is much more complicated, because to obtain a critical vertex we must remove $n, 1 \in \epsilon(e)$ twice (i.e. $v = e \setminus e_{an}e_{bn}e_{1x}e_{1y} \cdots$), so we are removing a nonconsecutive nondiagonal from e , which means $v \notin \text{Ker } \partial$. E.g. for $n=6$ we have $\dot{\mathcal{M}}_N = \dot{\mathcal{M}}_{N-1} = \dot{\mathcal{M}}_{N-2} = \{\}$, $\dot{\mathcal{M}}_{N-3} = \{e \setminus e_{14}e_{16}e_{36}\}$, $\dot{\mathcal{M}}_{N-4} = \{e \setminus e_{15}e_{16}e_{34}e_{56}, e \setminus e_{14}e_{16}e_{35}e_{56}, e \setminus e_{14}e_{16}e_{34}e_{46}, e \setminus e_{14}e_{15}e_{36}e_{56}, e \setminus e_{13}e_{16}e_{34}e_{36}, e \setminus e_{12}e_{16}e_{26}e_{34}, e \setminus e_{12}e_{16}e_{24}e_{36}, e \setminus e_{12}e_{14}e_{26}e_{36}\}$, but $H_{N-k}(\mathfrak{sol}_6; \mathbb{Z}_3) \cong 0$ for $k = 0, \dots, 5$.

Let $p \geq 5$ be prime and $n = \alpha p + \beta \in 2\mathbb{N}$ where $0 \leq \beta < p$. Over \mathbb{Z}_p the weights are $w_i \equiv \beta - 1, \beta - 3, \beta - 5, \dots, 3, 1, -1, -3, \dots, 5 - \beta, 3 - \beta, 1 - \beta$. Notice that the first p weights are exactly the permuted elements of \mathbb{Z}_p (by 3.9 below, we may assume $p < n$). Rings \mathbb{Z}_2 and \mathbb{Z}_3 are different than other \mathbb{Z}_p , since only in the former do we have just elements $0, \pm 1$ (such require only one removal per two weights). Over the latter, for any zero weight we have $|\mathbb{Z}_p \setminus \{0, \pm 1\}| = p - 3$ weights that require several removals, so $\dot{\mathcal{M}}_k = \{\}$ for $k \geq N - n/2$. This explains why in the table, summands \mathbb{Z}_p with large p appear at the middle of columns. \square

3.5. Invertible Integers. Next we compute the free part of the table.

Proposition 3.9. *If $\mathbb{Q} \leq R$ or $\mathbb{Z}_p \leq R$ with prime $p \geq n$ or $p > \frac{k+1}{2}$, then*

$$H_k(\mathfrak{sol}_n) \cong H^k(\mathfrak{sol}_n) \cong R^{\binom{n}{k}}.$$

Proof. Every nonzero element of \mathbb{Q} and \mathbb{Z}_p is a unit, so over \mathbb{Z}_p critical vertices are $\dot{\mathcal{M}} = \{v; \forall x \in \epsilon(v) : r_x - s_x \in p\mathbb{Z}\}$. For any vertex v and any index $x \in \epsilon(v)$ we have $|r_x - s_x| < n$, because only 1 and n can appear $n-1$ times as an index, all other indices appear fewer times. For any vertex $e_\sigma \neq v \in C_k$ there is some index $x \in \epsilon(v)$ with $0 < |r_x - s_x| \leq \frac{k+1}{2}$: if this is false and $a = \min \epsilon(v)$ and $b = \max \epsilon(v)$, then a and b appear at least $\frac{k+1}{2} =: m$ times each (only once can they appear together), so $m+m-1=k$ implies $v = e_{a_1 b} \dots e_{a_{m-1} b} e_{ab} e_{ab_1} \dots e_{ab_{m-1}}$ and thus $r_{a_1} - s_{a_1} = -1$, a contradiction. Thus under our assumptions, the only noninvertible weight $r_x - s_x$ is $0 \in \mathbb{Z}$, so $\dot{\mathcal{M}} = \{v; \forall x \in \epsilon(v) : r_x = s_x\} = \{e_\sigma\}$. The last equality holds, because if $\epsilon(v)$ is nonempty, then it contains a maximal element x (which can only appear on the right), but then $r_x > 0$ and $s_x = 0$. \square

3.6. Invertible Integers mod p . Now we are interested in the first occurrences of primes p that appear in the table as \mathbb{Z}_{p^l} for some l .

Proposition 3.10. *Let $\mathbb{Z}_p \leq R$ for some prime p . If $n = p+1$, then $H_k(\mathfrak{sol}_n) \cong H^k(\mathfrak{sol}_n) \cong R^{\binom{n}{k}} \oplus R^{\binom{n}{k-2n+3}}$. If $k = 2p-1$, then $H_k(\mathfrak{sol}_n) \cong R^{\binom{n}{k}} \oplus R^{\binom{n-p+1}{2}}$.*

Thus in the table, the first column where \mathbb{Z}_{p^l} (for some l) appears is the $p+1$ -th column, and the first row in which \mathbb{Z}_{p^l} (for some l) appears is the $2p-1$ -th row. We also know in what quantity the homology of that first column/row appears.

Proof. If $n = p+1$, then in a critical vertex v every weight is a multiple of p , which is so large that necessarily $\min \epsilon(v) = 1$ and $\max \epsilon(v) = n$, hence $\dot{\mathcal{M}} = \{e_\sigma, e_\sigma e_{12} e_{13} e_{14} \dots e_{1n} e_{2n} e_{3n} \dots e_{n-1, n}\}$ and $\dot{\partial} = 0$, so the result is apparent.

If $k = 2p-1$, then in a critical vertex v the index $\min \epsilon(v)$ appears only on the left (p times) and $\max \epsilon(v)$ appears only on the right (p times), so

$$\begin{aligned} \dot{\mathcal{M}}_{k-1} &= \emptyset \quad \text{and} \quad \dot{\mathcal{M}}_k = \{e_\sigma, e_{ab_1} \dots e_{ab_p} e_{b_1 b_p} \dots e_{b_{p-1} b_p}\} \quad \text{and} \\ \dot{\mathcal{M}}_{k+1} &= \{e_\sigma, e_{\{x\}} e_{ab_1} \dots e_{ab_p} e_{b_1 b_p} \dots e_{b_{p-1} b_p}, e_{ab_1} \dots e_{ab_p} e_{b_1 c} \dots e_{b_p c}, \dots\}, \end{aligned}$$

where $a < b_1 < \dots < b_p < c$. Now $\partial_k = 0$, but $\partial_{k+1} \neq 0$. Denote $b' = b+1$. We define

$$\mathcal{M}' = \left\{ \begin{array}{l} e_{ab_1} \dots e_{ab_{i-1}} e_{b'_{i-1} b_i} \dots e_{ab_p} e_{b_1 b_p} \dots e_{b_{p-1} b_p} \quad i < p \text{ is minimal} \\ e_{ab_1} \dots e_{ab_i} \dots e_{ab_p} e_{b_1 b_p} \dots e_{b_{p-1} b_p} \quad ; \text{ such that } b'_{i-1} \neq b_i \end{array} \right\},$$

i.e. we take the first nonconsecutive index and split it to obtain a consecutive one.

For every $u \xrightarrow{\mathcal{M}'} v$ there is exactly one $u \rightarrow v'$ (so zig-zags are unique), because of e_{ab_p} . Without the condition $i < p$, there would exist cycles in $\Gamma_{C_*}^{\mathcal{M}'}$, e.g. if $n = k = 3$, $p = 2$,

$$\begin{array}{ccc} e_{12} e_{14} e_{24} & \xleftarrow{\mathcal{M}'} & e_{12} e_{13} e_{24} e_{34} \\ & \xleftarrow{\mathcal{M}'} & \\ e_{13} e_{14} e_{34} & \xleftarrow{\mathcal{M}'} & e_{12} e_{14} e_{23} e_{34}. \end{array}$$

Our set \mathcal{M}' is a Morse matching, with surviving vertices

$$\dot{\mathcal{M}}'_k = \{e_\sigma, e_{a, a+1} e_{a, a+2} \dots e_{a, a+p-1} e_{a, b_p} e_{a+1, b_p} e_{a+2, b_p} \dots e_{a+p-1, b_p}\} \quad \text{and} \quad \dot{\partial}'_{k+1} = 0.$$

Since $a, b_p \in [n]$ and there are $p-1$ elements inbetween, there are $\binom{n-p+1}{2}$ choices. \square

4. STRICTLY TRIANGULAR MATRICES $\mathfrak{nil}_n(R)$

In the digraph for \mathfrak{nil}_n , there are many more isolated vertices, but it turns out that a subcomplex determines the homology of \mathfrak{nil}_n over a field.

4.1. Shifted binomials. There are some patterns in the table from 3, with which the modules \mathbb{Z}_{p^m} appear. Take for example the fourth column. By the Universal Coefficient Theorem, $\dim H_k(\mathfrak{so}\mathfrak{l}_4; \mathbb{Z}_2) = 1, 4, 6, 7, 15, 26, 24, 11, 2, 0, 0$. If we subtract $\binom{4}{k} = 1, 4, 6, 4, 1$, we get $0, 0, 0, 3, 14, 26, 24, 11, 2, 0, 0$. If we subtract $3\binom{4}{k}$ shifted by 3, we get $0, 0, 0, 0, 2, 8, 12, 8, 2, 0, 0$. If we subtract $2\binom{4}{k}$ shifted by 4, we get $0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0$. Thus $\dim H_k(\mathfrak{so}\mathfrak{l}_4; \mathbb{Z}_2) = \binom{4}{k} + 3\binom{4}{k-3} + 2\binom{4}{k-4}$. Doing this for all small n and p produces a table for $\dim H_k(\mathfrak{so}\mathfrak{l}_n; \mathbb{Z}_p)$:

$n \setminus p$	2
3	$\binom{n}{k} + \binom{n}{k-3}$,
4	$\binom{n}{k} + 3\binom{n}{k-3} + 2\binom{n}{k-4}$,
5	$\binom{n}{k} + 6\binom{n}{k-3} + 5\binom{n}{k-4} + 5\binom{n}{k-6} + 6\binom{n}{k-7} + \binom{n}{k-10}$,
6	$\binom{n}{k} + 10\binom{n}{k-3} + 9\binom{n}{k-4} + 30\binom{n}{k-6} + 61\binom{n}{k-7} + 30\binom{n}{k-8} + 15\binom{n}{k-10} + 19\binom{n}{k-11} + 5\binom{n}{k-12}$,

$n \setminus p$	3	5
4	$\binom{n}{k} + \binom{n}{k-5}$,	
5	$\binom{n}{k} + 3\binom{n}{k-5} + \binom{n}{k-6} + \binom{n}{k-8}$,	
6	$\binom{n}{k} + 6\binom{n}{k-5} + 3\binom{n}{k-6} + 6\binom{n}{k-8} + 4\binom{n}{k-9}$,	$\binom{n}{k} + \binom{n}{k-9}$.

4.2. Tensor product. The above observation is not a coincidence. Given a prime p and $\mathfrak{g} \in \{\mathfrak{nil}_n(R), \mathfrak{so}\mathfrak{l}_n(R)\}$, let $C_{\star,p}(\mathfrak{g}; R)$ be the chain subcomplex of $C_{\star,p}(\mathfrak{g}; R)$ spanned by vertices $\{v; \forall x \in \epsilon(v); r_x - s_x \in p\mathbb{Z}\}$. Thus for $p=2$ our $C_{\star,p}(\mathfrak{nil}_n)$ is a subcomplex, spanned by vertices in which every index appears even many times, e.g. $C_{1,p}=0$, $C_{2,p}=0$, $C_{3,p}=\langle e_{ab}e_{ac}e_{bc} \rangle$, $C_{4,p}=\langle e_{ab}e_{ac}e_{bd}e_{cd}, e_{ab}e_{ad}e_{bc}e_{cd}, e_{ac}e_{ad}e_{bc}e_{bd} \rangle$, etc. Let $\mathfrak{dgn}_n = \{\sum_x r_x e_{xx}\} \leq \mathfrak{so}\mathfrak{l}_n$ be the Lie subalgebra of all diagonals.

Proposition 4.1. *If $\mathbb{Z}_p \leq R$, then $\mathring{C}_{\star}(\mathfrak{so}\mathfrak{l}_n; R) \cong C_{\star,p}(\mathfrak{nil}_n; R) \otimes C_{\star}(\mathfrak{dgn}_n; R)$ and*

$$\dim H_k(\mathfrak{so}\mathfrak{l}_n; \mathbb{Z}_p) = \sum_{i+j=k} \dim H_i(C_{\star,p}(\mathfrak{nil}_n; \mathbb{Z}_p)) \cdot \binom{n}{j}$$

and $\{p\text{-torsion of } C_{\star}(\mathfrak{so}\mathfrak{l}_n; \mathbb{Z})\} = \{p\text{-torsion of } C_{\star,p}(\mathfrak{so}\mathfrak{l}_n; \mathbb{Z})\}$.

Proof. By 3.10 we have $\mathring{\mathcal{M}} = \{v; \forall x \in \epsilon(v): r_x - s_x \in p\mathbb{Z}\}$ and $\mathring{\partial} = \partial$. By definition, the tensor product of chain complexes $(C_{\star}, \partial_{\star})$ and $(C'_{\star}, \partial'_{\star})$ is given by $(C_{\star} \otimes C'_{\star})_k = \bigoplus_{i+j=k} C_i \otimes C'_j$ and $\partial \otimes \partial'(x \otimes x') = (\partial x) \otimes x' + (-1)^{|x|} x \otimes (\partial' x')$. Our \mathfrak{dgn}_n has trivial brackets, so $C_{\star}(\mathfrak{dgn}_n)$ has zero boundaries. In $\mathring{C}_{\star}(\mathfrak{so}\mathfrak{l}_n)$ the boundary of each vertex is a sum of all possible brackets of nondiagonal elements, because diagonals contribute the summands with weights as coefficients, which are 0 over \mathbb{Z}_p . It follows that $\mathring{C}_{\star}(\mathfrak{so}\mathfrak{l}_n; R) \cong C_{\star,p}(\mathfrak{nil}_n; R) \otimes C_{\star}(\mathfrak{dgn}_n; R)$. Then Künneth's theorem provides the formula for homology of the tensor product.

The statement that the p -torsion in the homology of the chain complex for $\mathfrak{so}\mathfrak{l}_n(\mathbb{Z})$ equals the p -torsion in the homology of $C_{\star,p}(\mathfrak{so}\mathfrak{l}_n; \mathbb{Z})$ follows from the fact that $C_{\star,p}$ is a direct summand of C_{\star} and the Universal Coefficient theorem: if the

complement of $C_{*,p}$ contained any p -torsion, then tensoring with \mathbb{Z}_p would imply that the complement contributes to $\dim H_k(\mathfrak{sol}_n; \mathbb{Z}_p)$, a contradiction with \mathcal{M} . \square

4.3. p -complex. We've seen that chain complex $C_{*,p}(\mathfrak{nil}_n; \mathbb{Z}_p)$ determines the homology of \mathfrak{sol}_n over \mathbb{Z}_p . We wish to find some patterns for the former. Computer experiments reveal the homology of $C_{k,p}(\mathfrak{nil}_n; \mathbb{Z})$ for small n and $p=2, 3, 5$:

$k \backslash n$	2	3	4	5	6	7
1						
2						
3	\mathbb{Z}	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}^3 \oplus \mathbb{Z}_2^3$	$\mathbb{Z}^4 \oplus \mathbb{Z}_2^6$		$\mathbb{Z}^5 \oplus \mathbb{Z}_2^{10}$
4		\mathbb{Z}	\mathbb{Z}^2	\mathbb{Z}^3		\mathbb{Z}^4
5		\mathbb{Z}	$\mathbb{Z}^2 \oplus \mathbb{Z}_3$	$\mathbb{Z}^3 \oplus \mathbb{Z}_3^3$		$\mathbb{Z}^4 \oplus \mathbb{Z}_3^6$
6			$\mathbb{Z}^2 \oplus \mathbb{Z}_2^3$	$\mathbb{Z}^5 \oplus \mathbb{Z}_2^{24} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3^2$		$\mathbb{Z}^9 \oplus \mathbb{Z}_2^{73} \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_3^4$
7			\mathbb{Z}^3	$\mathbb{Z}^{10} \oplus \mathbb{Z}_2^{23} \oplus \mathbb{Z}_4^3 \oplus \mathbb{Z}_3^3$		$\mathbb{Z}^{19} \oplus \mathbb{Z}_2^{84} \oplus \mathbb{Z}_4^{11} \oplus \mathbb{Z}_3^{10} \oplus \mathbb{Z}_2^2$
8			\mathbb{Z}	$\mathbb{Z}^4 \oplus \mathbb{Z}_3^3 \oplus \mathbb{Z}_3^3$		$\mathbb{Z}^{10} \oplus \mathbb{Z}_2^5 \oplus \mathbb{Z}_4^3 \oplus \mathbb{Z}_3^5 \oplus \mathbb{Z}^5 \oplus \mathbb{Z}_3^6$
9				$\mathbb{Z} \oplus \mathbb{Z}$		$\mathbb{Z}^5 \oplus \mathbb{Z}_2^{72} \oplus \mathbb{Z}_4^3 \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}_5$
10				\mathbb{Z}	$\mathbb{Z}^4 \oplus \mathbb{Z}_2^9 \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	$\mathbb{Z}^{19} \oplus \mathbb{Z}_2^{176} \oplus \mathbb{Z}_4^{18} \oplus \mathbb{Z}_3^6 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}_2^5 \oplus \mathbb{Z}_3^7$
11					$\mathbb{Z}^4 \oplus \mathbb{Z}_2^4$	$\mathbb{Z}^{19} \oplus \mathbb{Z}_2^{72} \oplus \mathbb{Z}_4^3 \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z} \oplus \mathbb{Z}_2^6 \oplus \mathbb{Z}_3$
12					\mathbb{Z}	$\mathbb{Z}^5 \oplus \mathbb{Z}_2^5 \oplus \mathbb{Z}_4^3 \oplus \mathbb{Z}_3^5 \oplus \mathbb{Z}_2^5$
13						$\mathbb{Z}^{10} \oplus \mathbb{Z}_2^{84} \oplus \mathbb{Z}_4^{11} \oplus \mathbb{Z}_3^{10} \oplus \mathbb{Z}^4 \oplus \mathbb{Z}_3^7$
14						$\mathbb{Z}^{19} \oplus \mathbb{Z}_2^{73} \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_3^4 \oplus \mathbb{Z}^2$
15						$\mathbb{Z}^9 \oplus \mathbb{Z}_2^2$
16						$\mathbb{Z}^2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}$
17						$\mathbb{Z}^4 \oplus \mathbb{Z}_2^{10}$
18						\mathbb{Z}^5
19						\mathbb{Z}
20						
21						\mathbb{Z}

The presence of summand \mathbb{Z}_3 in $C_{*,2}$ and summand \mathbb{Z}_2 in $C_{*,3}$ shows that $\hat{C}_{*,p}(\mathfrak{sol}_n; \mathbb{Z}) \not\cong C_*(\mathfrak{dgn}_n; \mathbb{Z}) \otimes C_{*,p}(\mathfrak{nil}_n; \mathbb{Z})$. This also means that a matching on $C_{*,p}(\mathfrak{nil}_n; \mathbb{Z}_p)$ cannot induce $\hat{\partial}=0$ (because for \mathfrak{nil}_n all weights are ± 1 , so digraph Γ_{C_*} over \mathbb{Z} or \mathbb{Z}_p has the same edges, hence \mathcal{M} is a matching over \mathbb{Z} iff over \mathbb{Z}_p , but the presence of torsion in homology over \mathbb{Z} implies nonzero differential). Thus we expect the computation to be tricky. It is difficult to find a recursion or a generating function or a closed formula for $H_k(C_{*,p}(\mathfrak{nil}_n; \mathbb{Z}_p))$. The homology of the whole complex $C_*(\mathfrak{nil}_n; \mathbb{Z})$ will be studied in an upcoming article by the same authors.

5. POSET TRIANGULAR MATRICES $\mathfrak{gl}_n^{\preceq}(R)$

For any partial ordering \preceq on $[n]$, our Lie algebra $\mathfrak{g} = \mathfrak{gl}_n^{\preceq}(R)$ is a subalgebra of $\mathfrak{gl}_n(R)$ that admits an R -module basis $\{e_{ij}; 1 \leq i \leq j \leq n\}$. This is a large class (parametrized by all finite posets) of *solvable* Lie algebras, i.e. $\exists r: \mathfrak{g}^{(r)} = 0$ where the derived series is defined inductively by $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(r)} = [\mathfrak{g}^{(r-1)}, \mathfrak{g}^{(r-1)}]$.

A number of properties of \mathfrak{sol}_n generalize to $\mathfrak{gl}_n^{\preceq}$.

Proposition 5.1. $\{\text{isolated vertices of } \Gamma_{C_*}\} = \{e_\sigma; \sigma \subseteq [n]\}$ over $R = \mathbb{Z}$.

If we restrict the ordering \preceq from $[n]$ to $[n-1]$, we obtain the subalgebra $\mathfrak{gl}_{n-1}^{\preceq}$.

Proposition 5.2. $H_k(\mathfrak{gl}_n^{\prec}) \cong H_k(\mathfrak{gl}_{n-1}^{\prec}) \oplus \dots$ and $H^k(\mathfrak{gl}_n^{\prec}) \cong H^k(\mathfrak{gl}_{n-1}^{\prec}) \times \dots$

We can define the set of edges \mathcal{M} similarly as for $\mathfrak{so}\mathfrak{l}_n$, by using diagonals.

Proposition 5.3. \mathcal{M} is a Morse matching, with $\overset{\circ}{\partial} = \partial$ and

$$\overset{\circ}{\mathcal{M}} = \{v; \forall x \in \epsilon(v): r_x - s_x \in R \setminus R^\times\}.$$

The proof is basically the same as in 3.2. In fact, for any Lie algebra $\mathfrak{g} \leq \mathfrak{gl}_n(R)$ that admits an R -module basis \mathcal{B} with $\{e_{ii}; i \in [n]\} \subseteq \mathcal{B} \subseteq \{e_{ij}; i, j \in [n]\}$, the set

$$\mathcal{M} = \left\{ \begin{array}{l} e_{a_1 x} \dots e_{a_r x} e_{x x} e_{x b_1} \dots e_{x b_s} e_{c_1 d_1} \dots e_{c_t d_t} e_\sigma, \quad x \in \epsilon(v) \text{ is minimal such} \\ e_{a_1 x} \dots e_{a_r x} \widehat{e_{x x}} e_{x b_1} \dots e_{x b_s} e_{c_1 d_1} \dots e_{c_t d_t} e_\sigma, \quad \text{that } r-s \text{ is a unit of } R \end{array} \right\}$$

is a Morse matching, where new boundary operators are the restrictions of old boundary operators and critical vertices are the wedges in which every index x has noninvertible weight (=number of times x appears on the right minus number of times x appears on the left) in R , i.e.

$$\overset{\circ}{\partial} = \partial \quad \text{and} \quad \overset{\circ}{\mathcal{M}} = \{v; \forall x \in \epsilon(v): r_x - s_x \in R \setminus R^\times\}.$$

However, $\overset{\circ}{\mathcal{M}}$ for general Lie algebras (such as $\mathfrak{gl}_n(R)$) may be much larger than $\overset{\circ}{\mathcal{M}}$ for \mathfrak{gl}_n^{\prec} (i.e. many vertices which don't contribute to homology may survive), but a decrease in size for the chain complex by a factor of 10 or 10^2 or 10^3 (depending on how many integers of the base ring R are invertible) is still quite beneficial. Another example where this matching is useful is $\mathfrak{so}_n(R) = \{\mathbf{a} \in \mathfrak{gl}_n(R); \mathbf{a}^t = -\mathbf{a}\}$ when R has characteristic 2, since it admits a basis $\{e'_{ab}, e_{cc}; a < b\}$ where $e'_{ab} = e_{ab} - e_{ba}$ and brackets are given by $[e'_{ab}, e'_{cd}] = \delta_{bc}e'_{ad} + \delta_{ad}e'_{bc} - \delta_{bd}e'_{ac} - \delta_{ac}e'_{bd}$ and $[e'_{ab}, e_{cc}] = \delta_{bc}e'_{ac} + \delta_{ac}e'_{bc}$, so the diagonals can be used to define \mathcal{M} .

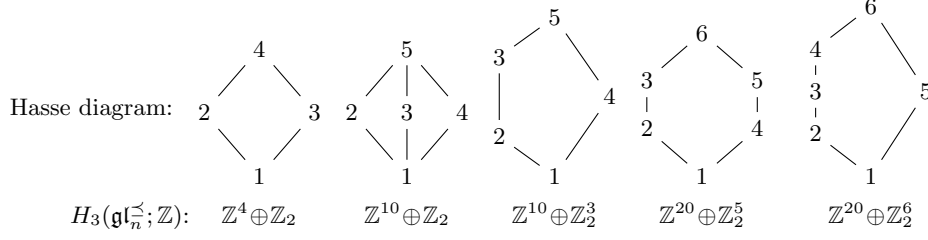
Remarks 5.4. Actually, this matching works in an even more general context: for an arbitrary Lie algebra (not necessarily a subalgebra of \mathfrak{gl}_n) all we need is a module basis \mathcal{B} and a subset $\{b_i\} \subseteq \mathcal{B}$ with the property that for every $b \in \mathcal{B}$ the bracket $[b_i, b]$ is either 0 or $\beta_i b$ for some unit $\beta_i \in R^\times$. This principle is not limited to Lie algebras, but to (co)homology theories whose (co)chain complex consists of exterior powers. It is somewhat analogous to the property [11, 6.5.3, p.178] and [7, 1.1.15, p.13] of group and Hochschild homology (where tensor powers are used instead of exterior powers), which say that removing all copies of the identity element 1 preserves the homology of the complex. Thus we call any such \mathcal{M} the *normalization matching*.

The most useful part for programmers is that we only need to remove columns and rows (very many of them) of boundary matrices ∂_k to obtain new boundary matrices $\overset{\circ}{\partial}_k$. This is in stark contrast to (co)homology theories where the weights in the digraph of the corresponding (co)chain complex are only 0 or ± 1 (such as simplicial or group or Hochschild (co)homology), since there the new boundary $\overset{\circ}{\partial}$ is almost never the restriction of the old boundary ∂ .

Proposition 5.5. If $\mathbb{Z} \cap R^\times = \{\pm 1\}$, then $H_1(\mathfrak{gl}_n^{\prec}) \cong H^1(\mathfrak{gl}_n^{\prec}) \cong R^n$, $H_2(\mathfrak{gl}_n^{\prec}) \cong H^2(\mathfrak{gl}_n^{\prec}) \cong R^{\binom{n}{2}}$, $H_3(\mathfrak{gl}_n^{\prec}) \cong R^{\binom{n}{3}} \oplus (\frac{R}{2R})^m$, where $m = |\{a, c \in [n]; \exists b: a \prec b \prec c\}|$.

This is obtained as in the proof of 3.7: $H_3(\mathfrak{g}) \cong \frac{\langle e_{\{a,b,c\}}, e_{ab}e_{ac}e_{bc} \rangle}{\langle e_{ac}e_{ad}e_{cd} - e_{ab}e_{ad}e_{bd}, 2e_{ab}e_{ac}e_{bc} \rangle}$, so the number of $\frac{R}{2R}$ summands equals the number of chains $a \prec b \prec c$ modulo the equivalence relation that identifies $a \prec b \prec d \sim a \prec c \prec d$, which is the number of pairs of elements in the poset that are comparable but not covering.

Example 5.6. A few drawings should illustrate the matter.



Proposition 5.7. *For every interval $[a, b] = \{a, x_1, \dots, x_t = b\}$ in poset $([n], \leq)$, the module $H_{2t-3}(\mathfrak{gl}_n^{\leq}; \mathbb{Z})$ has a direct summand \mathbb{Z}_t .*

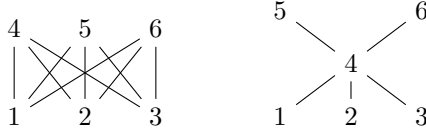
Indeed, $v = e_{ax_1} \dots e_{ax_t} e_{x_1 x_t} \dots e_{x_{t-1} x_t}$ is in $\text{Ker } \partial$, and no splitting is possible (because all x between a and b already appear in v), so vertex v is adjacent in the digraph only to $e_{aa}v$ and $e_{bb}v$, and the weights of those edges are $w_a = -t$ and $w_b = t$.

Theorem 5.8. *If $\mathbb{Q} \leq R$ or $\mathbb{Z}_p \leq R$ with prime $p \geq n$ or $p > \frac{k+1}{2}$, then*

$$H_k(\mathfrak{gl}_n^{\leq}(R)) \cong H^k(\mathfrak{gl}_n^{\leq}(R)) \cong R^{\binom{n}{k}}.$$

Using 5.3, our arguments are the same as in 3.9, we only need to remember that $\epsilon(v) \subseteq [n]$ does not have the usual total order but \leq instead. If every index $x \in \epsilon(v)$ has weight 0, then $v = e_\sigma$ is the wedge of diagonals, because maximal elements of $\epsilon(v)$ appear only on the right (i.e. $r_x > 0 = s_x$) and minimal elements appear only on the left (i.e. $s_x > 0 = r_x$).

Remarks 5.9. At first we thought that the above statement also holds when $p \geq \max_{a,b} |[a, b]|$, the size of the largest interval in the poset. We thought that the largest interval $[a, b] = \{a, x_1, \dots, x_t = b\}$ induced the vertex $e_{ax_1} \dots e_{ax_t} e_{x_1 x_t} \dots e_{x_{t-1} x_t}$ that would give the largest torsion \mathbb{Z}_t . This turned out to be false. Consider the following two posets, specified by their Hasse diagrams:



In the left poset, let $e = e_{14}e_{15}e_{16}e_{24}e_{25}e_{26}e_{34}e_{35}e_{36}$ be the wedge of all nondiagonals. Then $\partial(e) = 0$ and $\partial(e_{\{i\}}e) = \pm 3e$, so it generates \mathbb{Z}_3 , even though the largest interval has 2 elements. The analogous statement also holds for the right poset.

It remains an open problem to determine what the largest torsion that appears in $H_*(\mathfrak{gl}_n^{\leq})$ is, or to at least find some meaningful bounds on the torsion.

Theorem 5.10. *If $\mathbb{Q} \leq R$ or $\mathbb{Z}_p \leq R$ with prime $p \geq n$, then*

$$H^*(\mathfrak{gl}_n^{\leq}(R)) \cong \Lambda_R[x_1, \dots, x_n]$$

as graded algebras, where x_i has degree 1 and corresponds to matrix e_{ii} . Assuming $\mathbb{Z}_p \leq R$ with $p = n-1$, if $([n], \leq)$ has a least and greatest element a and b , then

$$H^*(\mathfrak{gl}_n^{\leq}(R)) \cong \Lambda_R[x_1, \dots, x_n, y]$$

as graded algebras, where y has degree $2p-1$ and corresponds to wedge $e_{ab} \bigwedge_x e_{ax} e_{xb}$, but if $([n], \leq)$ is not bounded, then $H^(\mathfrak{gl}_n^{\leq}(R)) \cong \Lambda_R[x_1, \dots, x_n]$.*

Proof. By 5.3, for $p \geq n$ the critical vertices $\mathring{\mathcal{M}}$ span a chain subcomplex of C_* , which is a direct summand whose complement is contractible. Thus under the assumptions, the inclusion of diagonals (which is a Lie algebra morphism) $\iota: \mathfrak{dgn}_n \rightarrow \mathfrak{gl}_n^{\leq}$ induces an isomorphism on all (co)homology modules, so the morphism of graded algebras $\iota^*: H^*(\mathfrak{gl}_n^{\leq}) \rightarrow H^*(\mathfrak{dgn}_n)$ is bijective. It remains to show that $H^*(\mathfrak{dgn}_n)$ is the exterior polynomial algebra. Since in \mathfrak{dgn}_n all brackets are zero and it has an R -module basis e_{11}, \dots, e_{nn} , the module $H^k(\mathfrak{g})$ has a basis $\{\chi_\sigma; \sigma \subseteq [n], |\sigma| = k\}$ where map χ_σ sends the basis vector e_σ to 1 and all other basis vectors e_τ to 0. By the formula from 2.1 for the cup product, we have $\chi_\sigma \smile \chi_\tau = \begin{cases} \pm \chi_{\sigma \cup \tau} & \text{if } \sigma \cap \tau = \emptyset \\ 0 & \text{if } \sigma \cap \tau \neq \emptyset \end{cases}$ where the sign is determined by the number of transpositions that are required to order $\sigma \cup \tau$. This is precisely the multiplication in $\Lambda_R[x_1, \dots, x_n]$.

Now assume that $p \geq n-1$. Critical vertices are wedges in which every index has weight 0 or p (only possible for a greatest element) or $-p$ (only possible for a least element). If the poset is not bounded, then critical vertices are precisely $\{e_\sigma; \sigma \subseteq [n]\}$, and the result follows as above. But if our poset equals the interval $[a, b]$, then $\mathring{\mathcal{M}} = \{e_\sigma, e'_\sigma; \sigma \in [n]\}$ and $\partial = 0$, where e'_σ is the wedge of e_σ and the vertex from 5.7. Thus $H^*(\mathfrak{gl}_n^{\leq})$ has a dual module basis $\{\chi_\sigma, \chi'_\sigma; \sigma \in [n]\}$. Furthermore, $\chi_\sigma \smile \chi'_\tau = \begin{cases} \pm \chi'_{\sigma \cup \tau} & \text{if } \sigma \cap \tau = \emptyset \\ 0 & \text{if } \sigma \cap \tau \neq \emptyset \end{cases}$ and $\chi'_\sigma \smile \chi'_\tau = 0$, so χ'_\emptyset is the additional generator y . \square

6. POSET STRICTLY TRIANGULAR MATRICES $\mathfrak{gl}_n^{\leq}(R)$

Analogously as in 4.3, we can define a chain subcomplex, the p -complex for $\mathfrak{gl}_n^{\leq}(R)$ and $\mathfrak{gl}_n^{\leq}(R)$, which begets:

Proposition 6.1. *If $\mathbb{Z}_p \leq R$, then $\mathring{C}_*(\mathfrak{gl}_n^{\leq}; R) \cong C_{*,p}(\mathfrak{gl}_n^{\leq}; R) \otimes C_*(\mathfrak{dgn}_n; R)$ and*

$$\dim H_k(\mathfrak{gl}_n^{\leq}; \mathbb{Z}_p) = \sum_{i+j=k} \dim H_i(C_{*,p}(\mathfrak{gl}_n^{\leq}; \mathbb{Z}_p)) \cdot \binom{n}{j}$$

and $\{p\text{-torsion of } C_(\mathfrak{gl}_n^{\leq}; \mathbb{Z})\} = \{p\text{-torsion of } C_{*,p}(\mathfrak{gl}_n^{\leq}; \mathbb{Z})\}$.*

The homology of Lie algebra $\mathfrak{gl}_m^{\leq}(\mathbb{Z})$ for small $m = 2^n$ with respect to the poset of all subsets $(2^{[n]}, \subseteq)$ has been computed in [1, p.203]. For general posets, some work on $H_*(\mathfrak{gl}_n^{\leq}(\mathbb{C}))$ has been done in [3].

Remarks 6.2. The homology of $C_{*,p}(\mathfrak{gl}_n^{\leq}; \mathbb{Z})$ can also contain q -torsion for $q \neq p$. Indeed, for every interval in $([n], \preceq)$ of size $qp+1$, the wedge from 5.7 has weights in $p\mathbb{Z}$, so it critical and it contributes $\mathbb{Z}_{pq} \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$.

7. AFTERWORD

7.1. Conclusion. We saw that there are easily definable matchings on difficult chain complexes, that give surprising insights into the homology table as well as the structure of the chain complex. The benefits are theoretical and also computational. For instance, the first five columns of the table in 3 for \mathfrak{sol}_n were computed by brute force, but the sixth required the use of 4.1. For later columns, the use of that proposition is key for efficient computations.

Over ring \mathbb{Z} , the set of critical vertices is smaller than the set of all vertices by a factor of about 10 (because the integer ring has many elements and only two

units), but over \mathbb{Z}_p the gain is much bigger (since only one element is a nonunit), especially for large p . E.g. for the first six columns of the table 3, over \mathbb{Z}_2 we have $\frac{|\text{rank } C_\star|}{|\text{rank } \tilde{C}_\star|} \doteq 10^2$, over \mathbb{Z}_3 we get $\frac{|\text{rank } C_\star|}{|\text{rank } \tilde{C}_\star|} \doteq 10^3$, and over \mathbb{Z}_5 even $\frac{|\text{rank } C_\star|}{|\text{rank } \tilde{C}_\star|} \doteq 10^4$.

For \mathfrak{gl}_n^\prec the gain is even higher than for \mathfrak{sol}_n : if both have the same number of basis elements e_{ab} , then the former has more diagonals e_{aa} that give strict conditions for the critical vertices, so $\dot{\mathcal{M}}$ for the latter is larger.

We expect these kinds of arguments to be applicable for many other Lie algebra families, as well as other objects of various (co)homology theories.

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